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Boundary Functions and Sets of Asymptotic Values

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1. INTRODUCTION

This paper presents a technique for constructing functions meromorphic in the unit disk with specific preassigned asymptotic behavior at the boundary. The construction technique is of a more geometric nature than the method used by S. Kierst in constructing a function meromorphic in the unit disk with any prescribed analytic set as its set of asymptotic values [2]. As a result, not only can the set of asymptotic values be prescribed as any nonempty analytic set, but the nature of the asymptotic paths can also be prescribed as point, arc, or spiral paths. In the case where the asymptotic paths terminate at points on the circumference, even the number of asymptotic values at each point can be restricted to no more than two.

The construction technique, which is the essential part of the proof for each of the results mentioned above, is presented separately in Sections 3–5 because it is of independent interest with applicability to situations other than those mentioned here. The particular applications which inspired this technique are presented in Sections 6 and 7. Although the resulting functions could easily have been constructed in the entire plane, the unit disk was chosen because of the precision that could be obtained at each point on the boundary.

2. DEFINITIONS

Let C and D denote respectively the unit circle $\{|z| = 1\}$ and the open unit disk $\{|z| < 1\}$ in the complex plane. A simple continuous curve $\lambda : z(t)(0 \leqslant t < 1)$ contained in D is called a *boundary path* if $|z(t)| \to 1$ as

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$t \to 1$. The *end* of a boundary path λ is the intersection of the closure $\bar{\lambda}$ of λ with C. If λ is a boundary path whose end is a single point z, λ is called a *boundary path at* z or simply a *path at* z.

Let f be a function from D into the Riemann sphere Ω. If $\lambda : z(t)(0 \leqslant t < 1)$ is a boundary path and there exists an $a \in \Omega$ such that $\lim_{t \to 1} f(z(t)) = a$, then a is called an *asymptotic value of* f and λ is called an *asymptotic path of* f. More specifically, a is a *point* (*arc*) *asymptotic value of* f if the end of λ is a point (arc or all of C).

By a *spiral* we shall mean a boundary path with the property that arg $z(t) \to +\infty$ as $t \to 1$. An asymptotic value of f obtained along a spiral σ is called a *spiral asymptotic value*, and σ is called a *spiral asymptotic path.* Let $\Gamma_P(f)$, $\Gamma_A(f)$, and $\Gamma_S(f)$ denote the sets of all point, arc, and spiral asymptotic values of f respectively. Let $\Gamma(f)$ denote the set of all asymptotic values of f and let $\Gamma(f, z)$ be the set of all point asymptotic values on paths at z.

A function Φ defined from C into Ω is called a *boundary function* for f if $\Phi(e^{i\theta}) \in \Gamma(f, e^{i\theta})$ for all $\theta(0 \leqslant \theta < 2\pi)$. Let $\Phi(S) = \{\Phi(z) : z \in S\}$. The notation $|\, \Gamma(f, e^{i\theta})|$ will be used to denote the cardinality of the set $\Gamma(f, e^{i\theta})$.

3. The Skeleton

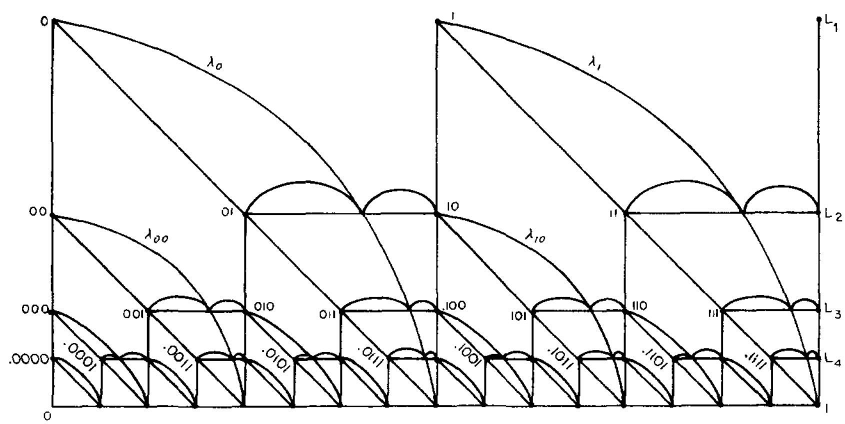

Figure 1

For simplicity we shall describe the construction in the rectangle

$$Q = \{z : 0 \leqslant \mathscr{R}(z) \leqslant 1, 0 \leqslant \mathscr{I}(z) \leqslant 1/2\} \text{ with } \mathscr{R}(z) = 0$$

identified with $\mathscr{R}(z) = 1$. Let $L_n = \{z : z \in Q, \mathscr{I}(z) = 2^{-n}\}$ $(n = 1, 2, \ldots)$.

Let $.i_1 \cdots i_n$ $(n = 1, 2,...; i_k = 0, 1)$ represent the point $(\sum_{k=1}^{n} i_k 2^{-k}, 2^{-n})$ on L_n. The points $.i_1 \cdots i_n$ $(n = 1, 2,...; i_k = 0, 1)$ will be called *vertices*, and L_n will be called the nth *level*. Note that L_n contains 2^n vertices.

Connect the vertices in the following manner:

(1) Join $.i_1 \cdots i_n$ to both $.i_1 \cdots i_n 0$ and $.i_1 \cdots i_n 1$ with rectilinear segments $(n = 1, 2,...; i_k = 0, 1)$.

(2) (a) Join .0 to the point $(1/2, 0)$ on the real axis with a curve having decreasing imaginary part and contained in the interior of the triangle with vertices .0, .1, and $(1/2, 0)$. Join .1 to the point $(1, 0)$ on the real axis in a similar manner (see Fig. 1).

(b) For $n > 1$, join the point $.i_1 \cdots i_{n-1} 0$ to the point

$$\left(\sum_{k=1}^{n-1} i_k 2^{-k} + 2^{-n}, 0\right)$$

on the real axis with a curve having decreasing imaginary part and contained in the interior of the triangle with vertices $.i_1 \cdots i_{n-1}0$, $.i_1 \cdots i_{n-1}1$, and

$$\left(\sum_{k=1}^{n-1} i_k 2^{-k} + 2^{-n}, 0\right).$$

We shall denote the curves described in (2) as λ-curves. When greater precision is needed, we shall denote the curve originating at $.i_1 \cdots i_n$ by $\lambda_{i_1 \cdots i_n}$. Thus λ_0 and λ_1 originate on L_1; but for $n > 1$, there are 2^{n-1} λ-curves originating on L_n (see Fig. 1).

(3) For each λ-curve originating on L_n, call it λ^n, let

$$d_k(\lambda^n) = \lambda^n \cap L_{n+k} \ (k = 1, 2,...).$$

Let $d_k^-(\lambda^n)$ be the first vertex on L_{n+k} which is less than $d_k(\lambda^n)$ (i.e., with respect to the linear ordering on L_{n+k}), and let $d_k^+(\lambda^n)$ be the first vertex on L_{n+k} greater than $d_k(\lambda^n)$ (e.g., if $\lambda^n = \lambda_{i_1 \cdots i_{n-1} 0}$, then $d_1^-(\lambda^n) = .i_1 \cdots i_{n-1}01$ and $d_1^+(\lambda^n) = .i_1 \cdots i_{n-1}10$). Join $d_k^-(\lambda^n)$ and $d_k(\lambda^n)$ with a rectilinear segment and also with a Jordan arc that is between L_{n+k-1} and L_{n+k} and does not cut any of the lines constructed in (1) and (2). These two arcs together form a Jordan curve containing both $d_k^-(\lambda^n)$ and $d_k(\lambda^n)$. Connect $d_k(\lambda^n)$ and $d_k^+(\lambda^n)$ in a similar manner (see Fig. 1).

This completes the construction of the skeleton which will be denoted by S^*. Let S denote the portion constructed in (1) and (2) only.

4. Sewing a Boundary Function onto S^*

We shall be concerned with constructing a meromorphic function that has a given function Φ as a boundary function. To this end it is sufficient to assume that if $\infty \in \Phi(C)$ then ∞ is not an isolated point in $\Phi(C)$. If this is not the case, let $A_\zeta^* = \{1/(a - \zeta) : a \in \Phi(C)\}$ for $\zeta \notin \Phi(C)$. Suppose f^* can be constructed meromorphic in D with a boundary function Φ^* such that $\Phi^*(C) = A_\zeta^*$. Then $f = (1/f^*) + \zeta$ is meromorphic in D and has the original Φ as a boundary function.

We shall further be concerned with boundary functions that have analytic subsets of Ω as their image sets. For this purpose it is sufficient to consider functions defined on the half-open interval $(0, 1]$ which are continuous on the left [3, p. 169]. Therefore, let Φ be a function defined on $(0, 1]$ which is continuous on the left and takes values in Ω. If Φ takes on the value ∞, we may assume there is a sequence of points $\{a_n\}_{n=1}^{\infty} \subset \Phi((0, 1])$ such that $|a_n| < |a_{n+1}|$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} a_n = \infty$. Extend Φ to the point 0 by defining $\Phi(0) = \Phi(1)$.

For each point $.i_1 \cdots i_n \in S$, define

$$g(.\, i_1 \cdots i_n) = \begin{cases} a_n & \text{if } \Phi\left(\sum_{k=1}^{n} i_k 2^{-k}\right) = \infty \\ \Phi\left(\sum_{k=1}^{n} i_k 2^{-k}\right) & \text{otherwise.} \end{cases}$$

Extend g to all of S in the following manner:

(1) g is constant on any segment whose end points have the same image value;

(2) On any segment whose vertices have the values a_n and a_{n+1} $(n = 1, 2, \ldots)$, let g be a homeomorphism onto a curve joining a_n to a_{n+1} and contained in the annulus $\{z : |a_n| < |z| < |a_{n+1}|\}$.

(3) In all other cases let g be a homeomorphism from the rectilinear segment joining two vertices to the rectilinear segment joining the images of the two vertices.

Define g on $\lambda_{i_1 \cdots i_n}$ constantly equal to $g(.i_1 \cdots i_n)$. For each λ^n, define g on the Jordan curve containing $d_k^-(\lambda^n)$ and $d_k(\lambda^n)$ in the following manner:

(1) if $g(d_k^-(\lambda^n)) = g(d_k(\lambda^n))$, g is constant on the entire Jordan curve.

(2) if $g(d_k^-(\lambda^n)) \neq g(d_k(\lambda^n))$, g is a homeomorphism from the Jordan

curve containing $d_k^-(\lambda^n)$ and $d_k(\lambda^n)$ to the circle containing $g(d_k^-(\lambda^n))$ and $g(d_k(\lambda^n))$ and having diameter equal to $| g(d_k^-(\lambda^n)) - g(d_k(\lambda^n))|$.

Define g on the Jordan curve containing $d_k(\lambda^n)$ and $d_k^+(\lambda^n)$ in a similar manner. In this way, g is defined and continuous on all of S^*.

5. Asymptotic Properties of g

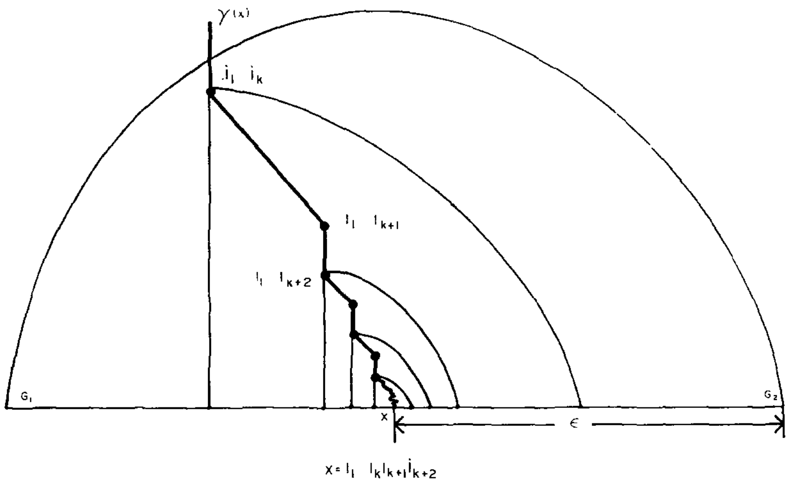

Figure 2

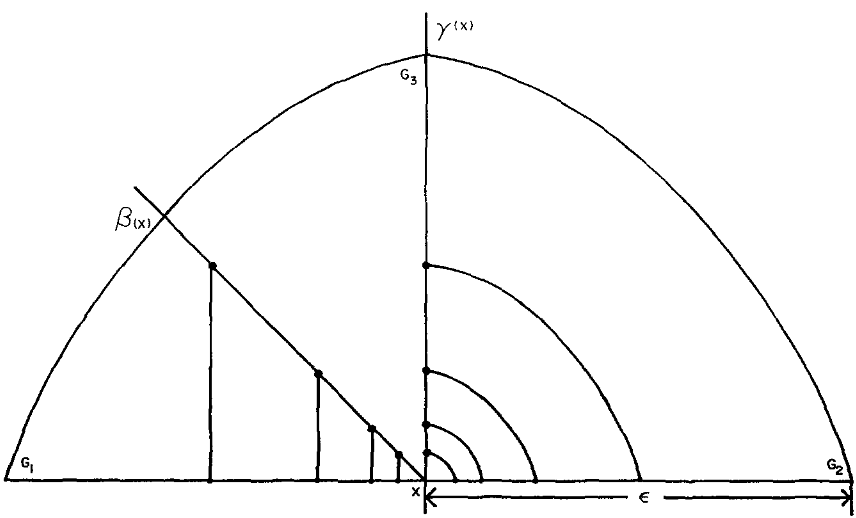

Figure 3

Each point $x \in [0, 1]$ that is not a dyadic rational can be expressed uniquely as

$$\sum_{k=1}^{\infty} i_k 2^{-k} \qquad (i_k = 0, 1).$$

Let $\gamma(x)$ denote the boundary path at x that is the union of the line segments joining the vertices $.i_1 \to .i_1 i_2 \to \cdots \to .i_1 \cdots i_n \to \cdots$. The path $\gamma(x)$ is a subset of S (see Fig. 2). For x a dyadic rational in $[0, 1]$, let $\gamma(x)$ be the intersection of S with the line $\mathscr{R}(z) = x$. Let $\beta(x)$ denote the intersection of S with the line containing x and having slope -1 (see Fig. 3).

Assume that g is defined on all of Q and has the given values on S^*. It follows from the left continuity of Φ and the manner in which Φ is woven onto S that $\gamma(x)$ is an asymptotic path of g at x with asymptotic value $\Phi(x)$ for each $x \in [0, 1]$. Similarly, if x is a dyadic rational in $[0, 1]$, $\beta(x)$ is an asymptotic path of g at x with asymptotic values $\Phi(x)$.

For each $x \in [0, 1]$ that is not a dyadic rational, there exists an $\epsilon > 0$ such that $N\epsilon(x) = \{z \in Q : |z - x| < \epsilon\}$ is divided into two components G_1 and G_2 by $\gamma(x)$ (see Fig. 2). Each vertex $.i_1 \cdots i_n \in \gamma(x)$ with $i_n = 0$ is the initial point of both a vertical line and a curved line (λ-curve) to the real axis. The vertical lines form a sequence of crosscuts in G_1 which converge to x. The curved lines form a sequence of crosscuts in G_2 which converge to x. Since the vertices on $\gamma(x)$ that have zero as last digit assume values that tend to $\Phi(x)$ as the vertices tend to x, $g(z) \to \Phi(x)$ uniformly on each of the sequences of crosscuts tending to x. As a result, $\Phi(x)$ is a cluster value on every boundary path at x. In particular, $\Phi(x)$ is the only asymptotic value on paths at x.

If x is a dyadic rational between 0 and 1, each sufficiently small neighborhood of x is divided into three components G_1, G_2, and G_3 by the boundary paths $\beta(x)$ and $\gamma(x)$ (see Fig. 3). An argument similar to the one above yields the conclusion that every boundary path at x which is contained in $G_1 \cup G_2$ will have $\Phi(x)$ as a cluster value. In particular, $\Phi(x)$ is the only asymptotic value on paths at x if boundary paths in G_3 are not considered.

There is, in fact, a λ-curve to x contained in G_3 on which g is constant (call the constant value α_λ). Although this value may not be $\Phi(x)$, it will always be in the set $\Phi((0, 1])$ since the value of g at each vertex is in $\Phi((0, 1])$. The definition of g on the Jordan curves linking λ to the boundary of G_3 limits any asymptotic value at x obtained on a path in G_3 to be either $\Phi(x)$ or α_λ.

We conclude that g has at least one asymptotic value at each point $x \in [0, 1]$ (i.e., $\Phi(x)$). At dyadic rational points, there is the possibility of at most one other asymptotic value (i.e., α_λ).

6. CONSTRUCTING FUNCTIONS WHICH HAVE Φ AS A BOUNDARY FUNCTION

Kierst [2] has constructed an example of a function meromorphic in D whose set of asymptotic values is any prescribed analytic subset of Ω. The following theorem presents a different construction of such a function. More control is obtained over the nature of the asymptotic paths and also over the asymptotic behavior of the function at each point on C.

THEOREM 1. *Let A be any nonempty analytic subset of Ω. There exists a function $f(z)$ meromorphic in D with the following asymptotic behavior*:

1. $A = \Gamma_P(f) = \Gamma(f)$;
2. *f has a boundary function Φ with $\Phi(C) = A$*;
3. *For all points $e^{i\theta} \in C$ $(0 \leqslant \theta < 2\pi)$, $1 \leqslant |\Gamma(f, e^{i\theta})| \leqslant 2$.*

Proof. Let Φ^* be the function defined from $(0, 1]$ onto A which is continuous on the left [3, p. 169]. By identifying the point 0 with the point 1, we can map C onto $(0, 1]$ with a homeomorphism h so that $\Phi = \Phi^* \circ h$ maps C onto A and is continuous in a counterclockwise direction. Let the circles $C_n = \{|z| = n/(n+1)\}$ replace the lines L_n $(n = 1, 2,...)$, and build the skeleton S^* in D making the obvious adjustments.

The function $g(z)$ which is continuous on S^* can then be approximated by a function $f(z)$ which is meromorphic in D and satisfies

$$\max_{z \in S^*} |f(z) - g(z)| \to 0$$

uniformly as $|z| \to 1$ [4]. The conclusions of the theorem have all been verified in Section 5.

Remark 1. If the circles $\{|z| = n\}$ replace the lines L_n $(n = 1, 2,...)$, the resulting function is meromorphic in the entire plane.

Remark 2. If $|A| = 1$, we can use the theorem of Walsh [5, p. 47, Theorem 15] to prescribe a pole at least in the interior of each of the Jordan curves in S^*. This would guarantee that $f(z)$ is not constant. In this case, Φ would be the only boundary function for f.

Remark 3. A consequence of the construction is that $|\Gamma(f, e^{i\theta})| = 2$ is possible only at the points on C which correspond (i.e., are the images under h) to dyadic rational points on $(0, 1]$.

The construction of a holomorphic function with Φ as a boundary function is possible if we use S instead of S^*. This results in a weakening of the limitation of $\Gamma(f)$ to A.

THEOREM 2. *Let A be any nonempty analytic subset of Ω. There exists a function $f(z)$ holomorphic in D with the following asymptotic behavior*:

1. $A \subset \Gamma_P(f) = \Gamma(f)$;
2. *f has a boundary function Φ with $\Phi(C) = A$.*

Proof. Define S as in the proof of Theorem 1. Since we are dealing with a holomorphic function here we must deal directly with the possibility that ∞ is isolated in A. To this end, redefine

$$g(.\, i_1 \cdots i_n) = \begin{cases} n & \text{if } \Phi\left(\sum_{k=1}^{n} i_k 2^{-k}\right) = \infty \\ \Phi\left(\sum_{k=1}^{n} i_k 2^{-k}\right) & \text{otherwise} \end{cases}$$

(compare with the definition in Section 4).

The remainder of the proof is the same as in Theorem 1 except for the use of a result of Arakeljan [1, p. 275, Theorem 2] which allows the conclusion that there exists a function f holomorphic in D and satisfying

$$\max_{z \in S} |f(z) - g(z)| \to 0$$

uniformly as $|z| \to 1$.

Remark 1. The conclusions of the theorem would be the same if all the λ-curves were excluded from S. The inclusion of these curves assures that $\Gamma(f, e^{i\theta}) = \{\Phi(e^{i\theta})\}$ at every point $e^{i\theta}$ which is the image under h of a point that is not a dyadic rational in $(0, 1]$.

Remark 2. To insure that f will not be constant, let $g(z) \to \infty$ as $|z| \to 1$ on λ_0. Unfortunately, in this case, this does not restrict the number of boundary functions for f even if $|A| = 1$.

Remark 3. If $|\Gamma(f, e^{i\theta})| > 1$, then $e^{i\theta}$ is the image (with respect to h) of a dyadic rational in $(0, 1]$. If $\Phi(e^{i\theta})$ and the α_λ at this point are finite and distinct, ∞ will also be an asymptotic value at $e^{i\theta}$. Thus $|\Gamma(f, e^{i\theta})| \geqslant 3$.

7. OTHER APPLICATIONS

For f meromorphic in D, Kierst [2] has characterized $\Gamma(f)$ as an analytic subset of Ω. The function constructed by Kierst for any analytic set containing more than one value is a normal function. It follows from this that

$\Gamma(f) = \Gamma_P(f)$. In [4] the author has constructed an example of a function meromorphic in D whose set of asymptotic values is any preassigned F_σ subset of Ω and satisfying $F_\sigma = \Gamma_s(f) = \Gamma(f)$. The following theorem generalizes that result and characterizes $\Gamma_s(f)$ as any analytic subset of Ω.

THEOREM 3. *Let A be any nonempty analytic subset of Ω. There exists a function $f(z)$ meromorphic in D with $A = \Gamma_s(f) = \Gamma(f)$.*

Proof. Define the vertices on the circles $C_n(n = 1, 2,...)$ as in the proof of Theorem 1. Rather than joining the points on C_n to those on C_{n+1} as before, use arcs whose arguments increase by at least 2π radians. The definition of g and the approximation by rational functions proceeds just as in Theorem 1. The new skeleton, however, contains only spirals as boundary paths and the conclusion follows.

THEOREM 4. *Let A be any nonempty analytic subset of Ω, and let δ satisfy $0 \leqslant \delta \leqslant 2\pi$. There exists a function f_δ meromorphic in D with the following asymptotic behavior*:

1. $A = \Gamma_A(f_\delta) = \Gamma(f_\delta)$
2. *For each $a \in A$ there exists a boundary path λ_a which is an asymptotis path for the value a and whose end is an arc of length δ.*

Proof. Define the vertices on the circles $C_n(n = 1, 2,...)$ as in Theorem 1. Rotate each circle $C_{2n}(n = 1, 2,...)$ through δ radians in a counterclockwise direction. Join the same vertices on C_n to those on C_{n+1} $(n = 1, 2,...)$ as in Theorem 1 but use arcs of the following type:

1. Increasing in modulus,
2. Contained in the annular region determined by C_n and C_{n+1},
3. Except for end points, do not meet the radii through the vertices to be connected.

The rest of the skeleton is completed in the obvious way. (The λ-curves described in (2) of Section 3 must now be constructed piecewise. For example, the first "piece" of λ_0 and λ_1 will terminate on C_2. When the vertices on C_2 are joined to those on C_3, λ_2 and λ_1 must also be extended to C_3 and so on). The construction of f_δ is completed as in the proof of Theorem 1.

For $a \in A$ there exists a point $e^{i\theta} \in C$ such that $\Phi(e^{i\theta}) = a$. The boundary path in this skeleton which corresponds to $\gamma(e^{i\theta})$ in the original skeleton (see Section 5) satisfies the requirements of the λ_a in the conclusion of this theorem. The end of λ_a is the arc on C with $e^{i\theta}$ and $e^{i(\theta+\delta)}$ as end points.

Remark. An analog to Theorem 2 is possible if we use S and Arakeljan's theorem [1, p. 275, Theorem 2]. The result is that f_δ is holomorphic in D, conclusion 2 holds, but $A \subset \Gamma_A(f_\delta) = \Gamma(f_\delta)$.

Two questions remain open:

1. If $\infty \in A$, can condition 1 in Theorem 2 be improved to read

$$A = \Gamma(f) = \Gamma_P(f)?$$

2. Can the upper bound on $|\Gamma(f, e^{i\theta})|$ in condition 3 of Theorem 1 be lowered to 1 for all θ or for all but a finite number of θ? Another way of considering this second problem is to try to reduce the number of boundary functions for f from the present countable number to a finite number. The best possible result, of course, is to construct f so that Φ is the only boundary function.

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